

# Classification of third-order symmetric Lorentzian manifolds

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**ABSTRACT.** Third-order symmetric Lorentzian manifolds, i.e. Lorentzian manifold with zero third derivative of the curvature tensor, are classified. These manifolds are exhausted by a special type of pp-waves, they generalize Cahen-Wallach spaces and second-order symmetric Lorentzian spaces.

**Keywords:** third-order symmetric Lorentzian manifold; pp-wave; holonomy algebra; curvature tensor

**MSC codes:** 53C29; 53C35; 53C50

## 1. Introduction

Symmetric pseudo-Riemannian manifolds constitute an important class of spaces. A direct generalization of these manifolds is provided by the so-called  $k$ -th order symmetric pseudo-Riemannian spaces  $(M, g)$  satisfying the condition

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0,$$

where  $k \geq 1$  and  $R$  is the curvature tensor of  $(M, g)$ . For Riemannian manifolds, the condition  $\nabla^k R = 0$  implies  $\nabla R = 0$  [22]. On the other hand, there exist pseudo-Riemannian  $k$ -th order symmetric spaces with  $k \geq 2$ , see e.g. [15, 20].

The fundamental paper [20] by J. M. M. Senovilla starts detailed investigation of symmetric Lorentzian spaces of second and higher orders. It contains many interesting results about such manifolds and their potential physical applications, e.g. Penrose limit type constructions, usage of super-energy tensors, higher order Lagrangian theories, supergravity, vanishing of quantum fluctuations. In particular, it is proven there that any second-order symmetric Lorentzian space admits a parallel null vector field, and it is conjectured that this property holds for symmetric Lorentzian spaces of higher orders.

A classification of four-dimensional second-order symmetric Lorentzian spaces is obtained by O. F. Blanco, M. Sánchez, J. M. M. Senovilla in the paper [4]. The result is based on the Petrov classification of the conformal Weyl curvature tensors. In [2] D. V. Alekseevsky and the author classified second-order Lorentzian symmetric manifolds. For that, the methods of the holonomy theory were used. Since the tensor  $\nabla R$  is parallel, its values at points are annihilated by the holonomy algebra, this algebraic condition allowed to find the exact form of  $\nabla R$ ; then it was shown that the Weyl conformal curvature tensor is parallel and the results of A. Derdzinski and W. Roter [8] about such spaces were applied. An alternative proof that used solutions of some PDE systems was obtained by the authors of [4] in [5]. Now we know that a second-order Lorentzian symmetric manifold is locally a product of a locally symmetric Riemannian manifold and of a Lorentzian manifold with the metric

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + H(du)^2, \quad H = (H_{1ij}u + H_{0ij})x^i x^j,$$

where  $H_{1ij}$  and  $H_{0ij}$  are symmetric real matrices. In [5] it is shown also that a simply connected geodesically complete second-order Lorentzian symmetric manifold is a global product of a (possibly trivial) Riemannian symmetric manifold and of  $\mathbb{R}^{n+2}$  with the above metric.

This paper is motivated by the lectures of M. Sánchez and J. M. Senovilla [18, 19], where the problems of classification of the higher order, and first of all of the third order, symmetric Lorentzian manifolds are discussed. In the present paper we classify third-order Lorentzian symmetric spaces. The main result can be stated as follows.

**THEOREM 1.** *Let  $(M, g)$  be a locally indecomposable Lorentzian manifold of dimension  $n+2 \geq 4$ . Then  $(M, g)$  is third-order symmetric if and only if locally there exist coordinates  $v, x^1, \dots, x^n, u$  such that*

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + H(du)^2, \quad H = (H_{2ij}u^2 + H_{1ij}u + H_{0ij})x^i x^j,$$

where  $H_{2ij}$ ,  $H_{1ij}$  and  $H_{0ij}$  are symmetric real matrices, the matrix  $H_{2ij}$  is nonzero, and it can be assumed to be diagonal.

By the Wu Theorem [25], any Lorentzian manifold  $(M, g)$  is either locally indecomposable, or it is locally a product of a Riemannian manifold  $(M_1, g_1)$ , and of a locally indecomposable Lorentzian manifold  $(M_2, g_2)$ . The manifold  $(M, g)$  is third-order symmetric if and only if  $(M_1, g_1)$  is locally symmetric and  $(M_2, g_2)$  is third-order symmetric. Consequently, Theorem 1 provides the complete local classification of third-order symmetric Lorentzian manifolds.

For the proof of Theorem 1, we use extend the ideas from [2]. The assumption that a Lorentzian manifold  $(M, g)$  is third-order symmetric implies that the holonomy algebra of  $(M, g)$  at a point  $x \in M$  annihilates the tensor  $\nabla^2 R_x \neq 0$ . This allows to find that exact form of  $\nabla^2 R$ . Unlike [2] we do not use the Weyl tensor, but find other tricks that allow to show that the manifold is a pp-wave. Then the condition  $\nabla^3 R = 0$  and simple computations allow us to find the coordinate form of the metric.

In particular, we prove the following two theorems conjectured by J. M. M. Senovilla [4, 5, 18] for higher-order symmetric Lorentzian manifolds.

**THEOREM 2.** *Let  $(M, g)$  be Lorentzian manifold of dimension  $n+2$  with the holonomy algebra  $\mathfrak{so}(1, n+1)$  and such that  $\nabla^3 R = 0$ . Then  $(M, g)$  is locally symmetric.*

**THEOREM 3.** *Any simply connected third-order symmetric Lorentzian manifold  $(M, g)$  admits a parallel null vector field.*

In [5, 18] it is shown that if a complete simply-connected Lorentzian manifold  $(M, g)$  is locally isometric to the product of a Riemannian symmetric space and a Lorentzian space with the metric as in Theorem 1, then  $(M, g)$  is globally isometric to one of such products. This implies

**THEOREM 4.** *Let  $(M, g)$  be a simply connected geodesically complete third-order symmetric Lorentzian manifold. Then  $(M, g)$  is a product of a (possibly trivial) symmetric Riemannian manifold and of  $\mathbb{R}^{n+2}$  with the metric from Theorem 1.*

Finally we discuss possible extension of these results to the case of higher order symmetric Lorentzian manifolds.

## 2. Holonomy algebras of Lorentzian manifolds

We recall some basic facts about the holonomy groups of Lorentzian manifolds that can be found in [3, 9, 16]. Let  $(M, g)$  be a Lorentzian  $(n+2)$ -dimensional manifold and  $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$  be its holonomy algebra at a point  $x \in M$ , i.e. the Lie algebra of the holonomy group at that point. Denote the tangent space  $T_x M$  by  $V$  and the metric  $g_x$  simply by  $g$ .

The manifold  $(M, g)$  is locally indecomposable (i.e. locally is not a direct product of two pseudo-Riemannian manifolds) if and only if the holonomy algebra  $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$  is weakly irreducible, i.e. it does not preserve any proper nondegenerate subspace of the tangent space.

Any weakly irreducible holonomy algebra  $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$  different from the Lorentz Lie algebra  $\mathfrak{so}(1, n+1)$  preserves a null line  $\mathbb{R}p$  of the tangent space. Denote by  $\mathfrak{sim}(n)$  the maximal subalgebra of  $\mathfrak{so}(1, n+1)$  preserving  $\mathbb{R}p$ . The Lie algebra  $\mathfrak{so}(1, n+1)$  will be identified with the space of bivectors  $\Lambda^2 V$  in such a way that

$$(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X, \quad X, Y, Z \in V.$$

Choose any null vector  $q \in V$  such that  $g(p, q) = 1$ . Let  $E \subset V$  be the Euclidean subspace orthogonal to  $p$  and  $q$ . We get the decomposition

$$(2.1) \quad V = \mathbb{R}p \oplus E \oplus \mathbb{R}q.$$

Let  $e_1, \dots, e_n$  be an orthonormal basis in  $E$ . We get the decomposition into the direct sum of vector subspaces

$$\mathfrak{sim}(n) = \mathbb{R}p \wedge q + \mathfrak{so}(n) + p \wedge E,$$

where  $\mathfrak{so}(n) = \mathfrak{so}(E) = \Lambda^2 E$ . If  $\mathfrak{g} \subset \mathfrak{sim}(n)$  is an arbitrary subalgebra, then the  $\mathfrak{so}(n)$ -projection of  $\mathfrak{g}$  is called *the orthogonal part of  $\mathfrak{g}$* . Decomposition (2.1) is a  $|1|$ -grading of  $V$  with the grading element  $p \wedge q$ .

The weakly irreducible Lorentzian holonomy algebras  $\mathfrak{g} \subset \mathfrak{sim}(n)$  are the following:

$$\begin{aligned} &(\text{type 1}) \quad \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E, & (\text{type 2}) \quad \mathfrak{h} + p \wedge E, \\ &(\text{type 3}) \quad \{\varphi(A)p \wedge q + A | A \in \mathfrak{h}\} + p \wedge E, & (\text{type 4}) \quad \{A + p \wedge \psi(A) | A \in \mathfrak{h}\} + p \wedge E, \end{aligned}$$

where  $\mathfrak{h} \subset \mathfrak{so}(n)$  is a Riemannian holonomy algebra;  $\varphi : \mathfrak{h} \rightarrow \mathbb{R}$  is a non-zero linear map that is zero on the commutant  $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ ; for the last algebra,  $E = E_1 \oplus E_2$  is an orthogonal decomposition,  $\mathfrak{h}$  annihilates  $E_2$ , i.e.  $\mathfrak{h} \subset \mathfrak{so}(E_1)$ , and  $\psi : \mathfrak{h} \rightarrow E_2$  is a surjective linear map that is zero on the commutant  $\mathfrak{h}'$ .

A locally indecomposable simply connected Lorentzian manifold admits a parallel null vector field if and only if its holonomy group is of type 2 or 4.

Let  $\mathfrak{g} \subset \mathfrak{sim}(n)$  be the holonomy algebra of the Lorentzian manifold  $(M, g)$  and  $\mathfrak{h} \subset \mathfrak{so}(E)$  be its orthogonal part. Then there exist the decompositions

$$(2.2) \quad E = E_0 \oplus E_1 \oplus \dots \oplus E_r, \quad \mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \dots \oplus \mathfrak{h}_r$$

such that  $\mathfrak{h}$  annihilates  $E_0$ ,  $\mathfrak{h}_i(E_j) = 0$  for  $i \neq j$ , and  $\mathfrak{h}_i \subset \mathfrak{so}(E_i)$  is an irreducible subalgebra for  $1 \leq i \leq r$ .

### 3. The holonomy algebra of a third-order symmetric Lorentzian manifold

A pseudo-Riemannian manifold  $(M, g)$  with the curvature tensor  $R$  is called a  $k$ -symmetric space if

$$\nabla^k R = 0, \quad \nabla^{k-1} R \neq 0.$$

So, one-symmetric spaces are the same as nonflat locally symmetric spaces ( $\nabla R = 0$ ,  $R \neq 0$ ).

Remark that for a Riemannian manifold the condition  $\nabla^k R = 0$  implies  $\nabla R = 0$  [22].

All indecomposable simply connected Lorentzian symmetric spaces are exhausted by the De Sitter and the anti De Sitter spaces and the Cahen-Wallach spaces. The last spaces have the commutative holonomy algebra  $p \wedge E$ .

Below we will prove the following

**THEOREM 5.** *The holonomy algebra of an  $(n+2)$ -dimensional locally indecomposable third-order symmetric Lorentzian manifold  $(M, g)$  is  $p \wedge E \subset \mathfrak{sim}(V)$ .*

It is known that any  $(n+2)$ -dimensional Lorentzian manifold with the holonomy algebra  $p \wedge E$  is a pp-wave (see e.g. [9, Sect 5.4]), i.e. locally there exist coordinates  $v, x^1, \dots, x^n, u$  such that the metric  $g$  can be written in the form

$$g = 2dvdu + \sum_i (dx^i)^2 + H(du)^2, \quad \partial_v H = 0.$$

We will need only to decide which functions  $H$  correspond to the third-order symmetric spaces.

### 4. Algebraic curvature tensors

For a subalgebra  $\mathfrak{g} \subset \mathfrak{so}(V)$  define *the space of algebraic curvature tensors of type  $\mathfrak{g}$* ,

$$\mathcal{R}(\mathfrak{g}) = \{R \in \Lambda^2 V^* \otimes \mathfrak{g} \mid R(u, v)w + R(v, w)u + R(w, u)v = 0 \text{ for all } u, v, w \in V\}.$$

If  $\mathfrak{g} \subset \mathfrak{so}(V)$  is the holonomy algebra of a manifold  $(M, g)$ , where  $V = T_x M$  is tangent space at some point  $x \in M$ , then the curvature tensor  $R_x$  of  $(M, g)$  belongs to  $\mathcal{R}(\mathfrak{g})$ . The spaces

$\mathcal{R}(\mathfrak{g})$  for holonomy algebras of Lorentzian manifolds are found in [11, 12]. For example, let  $\mathfrak{g} = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$ . For a subalgebra  $\mathfrak{h} \subset \mathfrak{so}(n)$  define the space

$$\mathcal{P}(\mathfrak{h}) = \{P \in E^* \otimes \mathfrak{h} \mid g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0 \text{ for all } X, Y, Z \in E\}.$$

Any  $R \in \mathcal{R}(\mathfrak{g})$  (considered as a tensor of type  $(4, 0)$ ) can be written as the sum

$$R = R_0 + P + T + v + L,$$

where

$$\begin{aligned} R_0 &= R_0^{ijkl}(e_i \wedge e_j) \odot (e_k \wedge e_l) \in \mathcal{R}(\mathfrak{h}), \\ P &= P^{ijk}(e_i \wedge e_j) \odot (p \wedge e_k), \quad P^{ijk} \in \mathbb{R}, \quad P(\cdot, q) \in \mathcal{P}(\mathfrak{h}), \quad P^{ijk} + P^{jki} + P^{kij} = 0, \\ T &= T^{ij}(p \wedge e_i) \odot (p \wedge e_j), \quad T^{ij} \in \mathbb{R}, \quad T^{ij} = T^{ji}, \\ v &= v^i(p \wedge q) \odot (p \wedge e_i), \quad v^i \in \mathbb{R}, \\ L &= \lambda(p \wedge q) \odot (p \wedge q), \quad \lambda \in \mathbb{R}. \end{aligned}$$

Here for bivectors  $\omega$  and  $\theta$ , we write

$$\omega \odot \theta = \omega \otimes \theta + \theta \otimes \omega.$$

The decomposition (2.2) implies

$$R_0 = R_{01} + \dots + R_{0r}, \quad P = P_1 + \dots + P_r, \quad R_{0\alpha} \in \mathcal{R}(\mathfrak{h}_\alpha), \quad P_\alpha(\cdot, q) \in \mathcal{P}(\mathfrak{h}_\alpha).$$

**Notation.** If  $S \in \otimes^r V \otimes \mathcal{R}(\mathfrak{g})$ , then we write

$$S = e_{a_1} \otimes \dots \otimes e_{a_r} \otimes R^{a_1 \dots a_r}, \quad R^{a_1 \dots a_r} \in \mathcal{R}(\mathfrak{g}),$$

where we assume that the indices take the values  $p, 1, \dots, n, q$ , and that  $e_p = p$ , and  $e_q = q$ . Next, we write

$$R^{a_1 \dots a_r} = R_0^{a_1 \dots a_r} + P^{a_1 \dots a_r} + T^{a_1 \dots a_r} + v^{a_1 \dots a_r} + L^{a_1 \dots a_r},$$

where e.g.

$$P^{a_1 \dots a_r} = P^{a_1 \dots a_r ijk}(e_i \wedge e_j) \odot (p \wedge e_k), \quad T^{a_1 \dots a_r} = T^{a_1 \dots a_r ij}(p \wedge e_i) \odot (p \wedge e_j).$$

Now we define the space of algebraic covariant derivatives of the curvature tensors

$$\nabla \mathcal{R}(\mathfrak{g}) = \{S \in V^* \otimes \mathcal{R}(\mathfrak{g}) \mid S_u(v, w) + S_v(w, u) + S_w(u, v) = 0 \text{ for all } u, v, w \in V\}.$$

If  $\mathfrak{g} \subset \mathfrak{so}(V)$  is the holonomy algebra of a manifold  $(M, g)$  at a point  $x \in M$ , then  $\nabla R_x \in \nabla \mathcal{R}(\mathfrak{g})$ . The decomposition of the space  $\nabla \mathcal{R}(\mathfrak{so}(r, s))$  into irreducible  $\mathfrak{so}(r, s)$ -modules is found in [21], see also [14].

Let us find the space  $\nabla \mathcal{R}(\mathfrak{g})$  for  $\mathfrak{g} = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$ .

**THEOREM 6.** Any  $S \in \nabla \mathcal{R}(\mathfrak{g})$  has the form

$$S = p \otimes R^p + e_t \otimes R^t + q \otimes R^q, \quad R^p, R^t, R^q \in \mathcal{R}(\mathfrak{g}),$$

and it holds

$$\begin{aligned} P^{pijk} &= T^{ijk} - T^{jik}, \quad P^{tijk} = 2R_0^{ijtk}, \quad R_0^q = 0, \quad P^q = 0, \\ e_t \otimes R_0^t &\in \nabla \mathcal{R}(\mathfrak{h}), \quad v^{ij} = 2T^{qij}, \quad v^{qi} = 2\lambda^i. \end{aligned}$$

**Proof.** We may write  $S = p \otimes R^p + e_t \otimes R^t + q \otimes R^q$  for some elements  $R^p, R^t, R^q \in \mathcal{R}(\mathfrak{g})$ . The equality

$$S_p(e_t, e_s) + S_{e_t}(e_s, p) + S_{e_s}(p, e_t) = 0$$

can be rewritten in the form

$$4R_0^{qijts} e_i \wedge e_j + 2P^{qtsk} p \wedge e_k = 0.$$

This implies  $R_0^q = 0$  and  $P^q = 0$ . Considering the vectors  $e_m, e_s, e_t$ , we get  $e_t \otimes R_0^t \in \nabla \mathcal{R}(\mathfrak{h})$ , and

$$P^{msrk} + P^{srnk} + P^{rnsk} = 0.$$

this means that  $P^{msrk}e_m \otimes e_k \otimes (e_s \wedge e_r) \in \mathcal{R}(\mathfrak{h})$ , and, in particular,  $P^{msrk} = -P^{ksrm}$ . Considering the vectors  $p, e_m, q$ , we get

$$-2T^{qim}p \wedge e_i - v^{qm}p \wedge q + 2\lambda^m p \wedge q + v^{mi}p \wedge e_i = 0.$$

Consequently,  $v^{mi} = 2T^{qim}$ , and  $v^{qm} = 2\lambda^m$ . Using the vectors  $q, e_t, e_s$ , we get the rest of the equalities.  $\square$

## 5. Proof of Theorem 2

Let  $(M, g)$  be an  $(n+2)$ -dimensional Lorentzian manifold with the holonomy algebra  $\mathfrak{g} = \mathfrak{so}(1, n+1)$  and the property  $\nabla^3 R = 0$ . It is noted in [4, 18] that using the methods from [22] it can be shown that

$$g(\nabla^2 R, \nabla^2 R) = 0.$$

The tensor  $\nabla^2 R$  is parallel and annihilated by the holonomy algebra. Consequently, for each point  $x \in M$ ,

$$\nabla^2 R_x : T_x M = V \rightarrow \nabla \mathcal{R}(\mathfrak{g}), \quad X \mapsto (\nabla_X \nabla R)_x$$

is an  $\mathfrak{g}$ -equivariant map. The multiplicity of the  $\mathfrak{g}$ -module  $V$  in the space  $\nabla \mathcal{R}(\mathfrak{g})$  is one [21, 14]. Consequently, the multiplicity of the trivial  $\mathfrak{g}$ -module  $\mathbb{R}$  in  $V \otimes \nabla \mathcal{R}(\mathfrak{g})$  is one as well, and  $\nabla^2 R_x$  belongs to this submodule. The extension of the metric  $g_x$  to the space  $V \otimes \nabla \mathcal{R}(\mathfrak{g})$  is non-degenerate and the space  $V \otimes \nabla \mathcal{R}(\mathfrak{g})$  can be decomposed into an orthogonal direct sum of  $\mathfrak{g}$ -invariant modules. Hence the restriction of  $g_x$  to  $\mathbb{R} \subset V \otimes \nabla \mathcal{R}(\mathfrak{g})$  is non-degenerate. We conclude that  $\nabla^2 R_x = 0$ , i.e.  $\nabla^2 R = 0$ . Results of [2, 20] imply that  $\nabla R = 0$ .  $\square$

## 6. Walker coordinates and a reduction lemma

Let  $(M, g)$  be a locally indecomposable third-order Lorentzian manifold with the weakly irreducible holonomy algebra  $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ . From Theorem 2 it follows that  $\mathfrak{g} \neq \mathfrak{so}(1, n+1)$ . Since  $\mathfrak{so}(1, n+1)$  is the only irreducible holonomy algebra [9], it follows that  $\mathfrak{g} \subset \mathfrak{sim}(n)$ .

Let  $(M, g)$  be a Lorentzian manifold with the holonomy algebra  $\mathfrak{g} \subset \mathfrak{sim}(n)$ . Then  $(M, g)$  admits (locally) a parallel distribution of null lines. According to [24], locally there exist the so called Walker coordinates  $v, x^1, \dots, x^n, u$  such that the metric  $g$  has the form

$$(6.1) \quad g = 2dvdu + h + 2Adu + H(du)^2,$$

where  $h = h_{ij}(x^1, \dots, x^n, u)dx^i dx^j$  is an  $u$ -dependent family of Riemannian metrics,  $A = A_i(x^1, \dots, x^n, u)dx^i$  is an  $u$ -dependent family of one-forms, and  $H = H(v, x^1, \dots, x^n, u)$  is a local function on  $M$ . Consider the local frame

$$(6.2) \quad p = \partial_v, \quad X_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2}H \partial_v.$$

Let  $E$  be the distribution generated by the vector fields  $X_1, \dots, X_n$ . Clearly, the vector fields  $p, q$  are null,  $g(p, q) = 1$ , the restriction of  $g$  to  $E$  is positive definite, and  $E$  is orthogonal to  $p$  and  $q$ . The vector field  $p$  defines the parallel distribution of null lines and it is recurrent, i.e.  $\nabla p = \theta \otimes p$ , where  $\theta = \frac{1}{2}\partial_v H du$ . Since the manifold is locally indecomposable, any other recurrent vector field is proportional to  $p$ . Next,  $p$  is proportional to a parallel vector field if and only if  $d\theta = 0$ , which is equivalent to  $\partial_v^2 H = \partial_i \partial_v H = 0$ . In the last case the coordinates can be chosen in such a way that  $\partial_v H = 0$  and  $\nabla p = \nabla \partial_v = 0$ , see e.g. [9].

Consider the metric (6.1), the vector fields (6.2) and an orthogonal frame  $e_1, \dots, e_n$  of the distribution  $E$ . Then the curvature tensor  $R$  of the metric and its covariant derivatives can be written as in Section 4 above with respect to the frame  $p, e_1, \dots, e_n, q$  and all coefficients being functions.

In [10] using results from [6] it is shown that there exist Walker coordinates

$$v, x_0 = (x_0^1, \dots, x_0^{n_0}), \dots, x_r = (x_r^1, \dots, x_r^{n_r}), u$$

adapted to the decomposition (2.2) and in addition with the property  $A = 0$ . This means that

$$h = h_0 + h_1 + \dots + h_r, \quad h_0 = \sum_{i=1}^{n_0} (dx_0^i)^2, \quad h_\alpha = \sum_{i,j=1}^{n_\alpha} h_{\alpha ij} dx_\alpha^i dx_\alpha^j, \quad \frac{\partial}{\partial x_\beta^k} h_{\alpha ij} = 0 \text{ if } \beta \neq \alpha.$$

For  $\alpha = 0, \dots, r$ , consider the submanifolds  $M_\alpha \subset M$  defined by  $x_\beta = c_\beta$ ,  $\alpha \neq \beta$ , where  $c_\beta$  are constant vectors. Then the induced metric is given by

$$g_\alpha = 2dvdu + h_\alpha + H_\alpha(du)^2.$$

The proof of the following lemma is the same as one of Lemma 1 from [2].

**LEMMA 1.** *The submanifold  $M_\alpha \subset M$  is totally geodesic. The orthogonal part of the holonomy algebra  $\mathfrak{g}_\alpha$  of the metrics  $g_\alpha$  coincides with  $\mathfrak{h}_\alpha \subset \mathfrak{so}(E_\alpha)$ , which is irreducible for  $\alpha = 1, \dots, r$ . If the metric  $g$  is third-order symmetric, then the curvature tensor of each metric  $g_\alpha$  satisfies  $\nabla^3 R = 0$ .*

Remark that the metric  $g_\alpha$  must not be indecomposable.

## 7. Proof of Theorem 3

We will show that there are no Lorentzian manifolds with the property  $\nabla^3 R = 0$  and with the holonomy algebras of type 1 or 3.

**LEMMA 2.** *Let  $\mathfrak{g} = \mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$  with no assumption on  $\mathfrak{h} \subset \mathfrak{so}(n)$ , then the subspace of  $V \otimes \nabla\mathcal{R}(\mathfrak{g})$  annihilated by  $\mathfrak{g}$  is trivial.*

**Proof.** Let  $S \in V \otimes \nabla\mathcal{R}(\mathfrak{g})$  and suppose that it is annihilated by  $\mathfrak{g}$ . Let us write

$$S = p \otimes S^p + e_t \otimes S^t + q \otimes S^q,$$

where each element  $S^p, S^t, S^q \in \nabla\mathcal{R}(\mathfrak{g})$  is as in Theorem 6, e.g.

$$S^p = p \otimes R^{pp} + e_t \otimes R^{pt} + q \otimes R^{pq}, \quad R^{pp}, R^{pt}, R^{pq} \in \mathcal{R}(\mathfrak{g}).$$

Note that

$$(p \wedge q)p = -p, \quad (p \wedge q)q = q, \quad (p \wedge q)e_i = 0.$$

Consequently, if

$$Q = Q^{a_1 \dots a_r} e_{a_1} \otimes \dots \otimes e_{a_r}, \quad Q^{a_1 \dots a_r} \in \mathbb{R}$$

is a tensor annihilated by  $p \wedge q$ , then  $Q^{a_1 \dots a_r} = 0$  whenever in  $Q^{a_1 \dots a_r}$  the number of indices equals to  $p$  is different from the number of indices equal to  $q$ . This implies that

$$S^p = q \otimes \lambda^{pq}(p \wedge q) \odot (p \wedge q).$$

The condition that  $p \wedge e_s \in \mathfrak{g}$  annihilates  $S$  implies the equations

$$(7.1) \quad S^s = (p \wedge e_s) \cdot S^p, \quad (p \wedge e_s) \cdot S^t = -\delta_{st} S^q, \quad (p \wedge e_s) \cdot S^q = 0.$$

Using this we obtain

$$\begin{aligned} S^s &= e_s \otimes \lambda^{pq}(p \wedge q) \odot (p \wedge q) + 2q \wedge \lambda^{pq}(p \wedge e_s) \odot (p \wedge q), \\ -\delta_{st} S^q &= (p \wedge e_t) \cdot S^s = -\delta_{st} p \otimes \lambda^{pq}(p \wedge q) \odot (p \wedge q) + 2e_s \otimes \lambda^{pq}(p \wedge e_t) \odot (p \wedge q) \\ &\quad + 2e_t \otimes \lambda^{pq}(p \wedge e_s) \odot (p \wedge q) + 2q \otimes \lambda^{pq}(p \wedge e_s) \odot (p \wedge e_t). \end{aligned}$$

Taking  $s \neq t$ , we get  $\lambda^{pq} = 0$ . Consequently,  $S = 0$ .  $\square$

**LEMMA 3.** *Let  $J$  be a complex structure on  $\mathbb{R}^{2m}$ . Then the eigenvalues of  $J$  on  $\odot^2 \mathbb{R}^{2m}$  are zero and pure imaginary, and the eigenvalues of  $J$  on  $\odot^3 \mathbb{R}^{2m}$  are pure imaginary.*

**Proof.** As usual, the complexification  $\mathbb{R}^{2m} \otimes \mathbb{C} = \mathbb{C}^{2m}$  can be decomposed as  $\mathbb{C}^{2m} = W \oplus \bar{W}$ , where  $W$  is the eigenspace of the extension of  $J$  with eigenvalue  $i$ , and  $\bar{W}$  is the eigenspace of the extension of  $J$  with eigenvalue  $-i$ . Next,

$$(7.2) \quad (\odot^2 \mathbb{R}^{2m}) \otimes \mathbb{C} = \odot^2 \mathbb{C}^{2m} = (\odot^2 W) \oplus (W \otimes \bar{W}) \oplus (\odot^2 \bar{W}).$$

This shows that the eigenvalues of  $J$  on  $\odot^2 \mathbb{R}^{2m}$  are  $2i$ ,  $0$  and  $-2i$ . Similarly,

$$(\odot^3 \mathbb{R}^{2m}) \otimes \mathbb{C} = \odot^3 \mathbb{C}^{2m} = (\odot^3 W) \oplus (\odot^2 W \otimes \bar{W}) \oplus (W \otimes \odot^2 \bar{W}) \oplus (\odot^3 \bar{W}),$$

i.e. the eigenvalues of  $J$  on  $\odot^2 \mathbb{R}^{2m}$  are  $3i$ ,  $i$ ,  $-i$  and  $-3i$ .  $\square$

**LEMMA 4.** *Let  $\mathfrak{g} \subset \mathfrak{sim}(n)$  be of type 3 with an irreducible orthogonal part  $\mathfrak{h} \subset \mathfrak{so}(n)$ , then the subspace of  $V \otimes \nabla\mathcal{R}(\mathfrak{g})$  annihilated by  $\mathfrak{g}$  is trivial.*

**Proof.** Since  $\mathfrak{g}$  is of type 3, and  $\mathfrak{h} \subset \mathfrak{so}(n)$  is irreducible, it holds  $\mathfrak{h} \subset \mathfrak{u}(m)$ ,  $n = 2m$ , and

$$\mathfrak{g} = \mathbb{R}(p \wedge q + cJ) + \mathfrak{h}' + p \wedge E, \quad c \in \mathbb{R}, \quad c \neq 0,$$

where  $J$  is the complex structure on  $\mathbb{R}^{2m}$ . Suppose that  $S \in V \otimes \nabla \mathcal{R}(\mathfrak{g})$  is annihilated by  $\mathfrak{g}$ . Let as in Lemma 2

$$S = p \otimes S^p + e_t \otimes S^t + q \otimes S^q, \quad S^p, S^t, S^q \in \nabla \mathcal{R}(\mathfrak{g}).$$

Since  $\mathfrak{g}$  is of type 3, it holds  $\lambda^{ab} = 0$ . Let  $\xi = p \wedge q + cJ \in \mathfrak{g}$ . It is clear that  $\xi \cdot (p \otimes S^p) = 0$ . Consequently,

$$S^p = \xi \cdot S^p = -p \otimes R^{pp} + p \otimes \xi \cdot R^{pp} + \xi \cdot (e_t \otimes R^{pt}) + q \otimes R^{pq} + q \otimes \xi \cdot R^{pq}.$$

We get the equations

$$\xi \cdot R^{pp} = 2R^{pp}, \quad \xi \cdot (e_t \otimes R^{pt}) = e_t \otimes R^{pt}, \quad \xi \cdot R^{pq} = 0.$$

Note that for each  $R \in \mathcal{R}(\mathfrak{g})$  it holds

$$R = R_0 + P + T + v, \quad \xi \cdot R = -P + cJP - 2T + cJT - v + cJv.$$

Using this we get

$$R_0^{pp} = 0, \quad cJP^{pp} = 3P^{pp}, \quad cJT^{pp} = 4T^{pp}, \quad cJv^{pp} = 3v^{pp},$$

$$cJ(e_t \otimes R_0^{pt}) = e_t \otimes R_0^{pt}, \quad cJ(e_t \otimes T^{pt}) = 3e_t \otimes T^{pt}.$$

Note that since  $\mathfrak{h} \subset \mathfrak{u}(m)$ , it holds  $JR_0 = 0$  for each  $R_0 \in \mathcal{R}(\mathfrak{h})$ , and  $JP = P^{ijk}(e_i \wedge e_j) \otimes Je_k$  for each  $P = P^{ijk}(e_i \wedge e_j) \otimes e_k \in \mathcal{P}(\mathfrak{h})$ . We conclude that

$$R_0^{pt} = R_0^{pp} = P^{pp} = T^{pp} = v^{pp} = 0,$$

in particular,  $R^{pp} = 0$  (the equality  $T^{pp} = 0$  follows from the previous lemma). From Theorem 6 it follows that  $P^{pt} = 0$ , and the tensor  $T^{pijk}$  is symmetric in  $i, j, k$ . From the equality  $cJ(e_t \otimes T^{pt}) = 3e_t \otimes T^{pt}$  and the previous lemma it follows that  $T^{pt} = 0$ . We get that  $R^{pt} = v^{pt}$ . The equality  $\xi \cdot S^{pq} = 0$  implies  $S^{pq} = 0$ . From this and Theorem 6 it follows that  $v^{pt} = 0$ . Thus,  $S^p = 0$ . From (7.1) it follows that  $S = 0$ .  $\square$

Let  $(M, g)$  be simply connected third-order symmetric Lorentzian manifold. Suppose that it does not admit a parallel null vector field. Then its holonomy algebra is either of type 1 or 3. Recall that there are no neither second-order symmetric nor locally symmetric Lorentzian manifolds with holonomy algebras of type 1 or 3. If the holonomy algebra is of type 1, then from Lemma 2 it follows that  $\nabla^2 R = 0$ , which gives a contradiction. Hence the holonomy algebra is of type 3. Similarly, from Lemma 4 it follows that the orthogonal part  $\mathfrak{h} \subset \mathfrak{so}(n)$  is not irreducible. Consider the metrics  $g_\alpha$  as in Section 6. Since  $\mathfrak{g}$  is of type 3, it holds  $\partial_v^2 H = 0$  and  $\partial_v \partial_{x_{i_\alpha}} H_\alpha \neq 0$  for some  $\alpha$  and  $i_\alpha$  [13]. This implies that the metric  $g_\alpha$  is indecomposable with  $\nabla^3 R = 0$  and its holonomy algebra is of type 3 with irreducible orthogonal part  $\mathfrak{h}_\alpha \subset \mathfrak{so}(n_\alpha)$ . From Lemma 4 it follows that  $\nabla^2 R = 0$ , this gives a contradiction.  $\square$

## 8. Proof of Theorem 5

Now we assume that  $(M, g)$  is an indecomposable third-order symmetric Lorentzian manifold with the holonomy algebra  $\mathfrak{g} \subset \mathfrak{sim}(n)$  annihilating the null vector  $p \in V$ . The metric  $g$  written as (6.1) than satisfies  $\partial_v H = 0$ . Clearly it holds  $\Gamma_{va}^b = 0$ ,  $R_{vabc} = 0$ , and  $\nabla R_{abcd;e} = 0$  whenever one of the indices is  $v$ . In particular, for any  $x \in M$  it holds  $\nabla^2 R_x \in (\mathbb{R}p \oplus E) \otimes \nabla \mathcal{R}(\mathfrak{g})$ .

**LEMMA 5.** *Let  $\mathfrak{g} = \mathfrak{h} + p \wedge E$  with  $\mathfrak{h} \subset \mathfrak{so}(n)$  being an arbitrary subalgebra, then any element  $S \in (\mathbb{R}p \oplus E) \otimes \nabla \mathcal{R}(\mathfrak{g})$  annihilated by  $\mathfrak{g}$  is of the form*

$$S = (T^{pij}p \otimes p + T^{pkij}p \otimes e_k - T^{pkij}e_k \otimes p) \otimes (p \wedge e_i) \odot (p \wedge e_j),$$

*such that the tensors  $T^{pij}e_i \otimes e_j$  and  $T^{pkij}e_k \otimes e_i \otimes e_j$  are symmetric and annihilated by  $\mathfrak{h}$ .*

**Proof.** Let  $S \in (\mathbb{R}p \oplus E) \otimes \nabla\mathcal{R}(\mathfrak{g})$  and suppose that it is annihilated by  $\mathfrak{g}$ . Let us write

$$S = p \otimes S^p + e_t \otimes S^t,$$

where the elements  $S^p, S^t \in \nabla\mathcal{R}(\mathfrak{g})$  are as in Theorem 6. Since  $\mathfrak{g}$  is of type 2, all tensors  $L$  and  $v$  are zero. Let  $A \in \mathfrak{h}$ . Then it is clear that  $A \cdot S^p = 0$ . Since

$$S^p = p \otimes (R_0^{pp} + P^{pp} + T^{pp}) + e_m \otimes (R_0^{pm} + P^{pm} + T^{pm}),$$

we conclude that

$$A \cdot P^{pp} = 0, \quad A \cdot T^{pp} = 0, \quad A \cdot (e_m \otimes R_0^{pm}) = 0, \quad A \cdot (e_m \otimes T^{pm}) = 0.$$

The spaces  $\mathcal{P}(\mathfrak{h})$  do not contain elements annihilated by  $\mathfrak{h}$  [12], consequently,  $P^{pp} = 0$ . From this and Theorem 6 it follows that the tensor  $T^{pmij}$  is symmetric. The spaces  $\mathcal{R}(\mathfrak{h})$  do not contain submodules isomorphic to  $E$  [1], this implies that  $e_m \otimes R_0^{pm} = 0$ , i.e.  $R_0^{pm} = 0$ .

The condition that  $p \wedge e_s \in \mathfrak{g}$  annihilates  $S$  implies the equations

$$(8.1) \quad S^s = (p \wedge e_s) \cdot S^p, \quad (p \wedge e_s) \cdot S^t = 0.$$

Using this we obtain

$$S^s = 4p \otimes R_0^{ppijks} (e_i \wedge e_j) \odot (p \wedge e_k) - p \otimes P^{ps} - p \otimes T^{ps} + 2e_t \otimes P^{ptisk} (p \wedge e_i) \odot (p \wedge e_k).$$

Next,

$$0 = (p \wedge e_m) \cdot S^s = p \otimes (8R_0^{ppimks} - 2P^{psimk} - 2P^{pmisk}) (p \wedge e_i) \odot (p \wedge e_k).$$

From Theorem 6 it follows that  $P^{pmisk} = 2R_0^{ppimsk}$ . This and the last equality imply that  $R_0^{pp} = 0$  and  $P^{pt} = 0$ . This proves the lemma.  $\square$

Now we continue the proof of Theorem 5.

Consider the metric (6.1), the vector fields (6.2) and an orthogonal frame  $e_1, \dots, e_n$  of the distribution  $E$ . Then  $\nabla^2 R$  can be written as in Lemma 5 with respect to the frame  $p, e_1, \dots, e_n, q$  and the elements  $T^{pij}$  and  $T^{pkij}$  being functions. Note that the 1-form  $du$  is dual to the vector field  $p$  and it is parallel.

Suppose that  $T^{pkij} = 0$ . This means that

$$\nabla^2 R = du \otimes du \otimes T^{pp},$$

where  $\nabla^2 R$  is considered as the tensor of type  $(4, 2)$ . Since  $\nabla^3 R = 0$  and  $\nabla du = 0$ , we get  $\nabla T^{pp} = 0$ . Consequently,

$$\nabla^2 R = \nabla^2 \frac{u^2}{2} T^{pp},$$

i.e.

$$\nabla \left( \nabla R - \nabla \frac{u^2}{2} T^{pp} \right) = 0.$$

We see that value of the tensor  $\nabla R - \nabla \frac{u^2}{2} T^{pp}$  at any point  $x \in M$  is annihilated by the holonomy algebra and belongs to the space  $\nabla\mathcal{R}(\mathfrak{g})$ . As in [2, Lemma 3] or as in Lemma 5 it can be shown that

$$\nabla R - \nabla \frac{u^2}{2} T^{pp} = du \otimes T_1, \quad T_1 = T_1^{ij} (p \wedge e_i) \odot (p \wedge e_j).$$

Consequently,

$$\nabla \left( R - \frac{u^2}{2} T^{pp} - u T_1 \right) = 0.$$

This easily implies that

$$R = \frac{u^2}{2} T^{pp} + u T_1 + T_0, \quad T_0 = T_0^{ij} (p \wedge e_i) \odot (p \wedge e_j),$$

i.e. the curvature tensor  $R$  is the same of a pp-wave. We conclude that the metric  $g$  is locally a pp-wave metric, and  $\mathfrak{g} = p \wedge E$ .

Suppose now that  $T^{pkij} \neq 0$  and suppose that  $\mathfrak{h} \neq 0$ . Consider two cases.

**Case 1.**  $\mathfrak{h} \subset \mathfrak{so}(n)$  is irreducible. The Riemannian holonomy algebra  $\mathfrak{h} \subset \mathfrak{so}(n)$  annihilates a symmetric 3-tensor. It is known [17] that this happens only for the irreducible representations of the Lie algebras  $\mathfrak{so}(3)$ ,  $\mathfrak{su}(3)$ ,  $\mathfrak{sp}(3)$ , and  $F_4$  in dimensions 5, 8, 14, and 26, respectively. We



do not need such a strong statement and we prove a weaker one in order to make the exposition more self-contained. Recall that a symmetric Berger algebra  $\mathfrak{h} \subset \mathfrak{so}(n)$  is the holonomy algebra of a symmetric Riemannian manifold different from  $\mathfrak{so}(n)$ ,  $\mathfrak{u}(\frac{n}{2})$  and  $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$ .

**LEMMA 6.** *If an irreducible Riemannian holonomy algebra  $\mathfrak{h} \subset \mathfrak{so}(n)$  admits a nonzero invariant symmetric 3-tensor, then  $\mathfrak{h} \subset \mathfrak{so}(n)$  is a symmetric Berger algebra.*

**Proof.** We must show, that the Riemannian holonomy algebras  $\mathfrak{so}(n)$ ,  $\mathfrak{u}(\frac{n}{2})$ ,  $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$ ,  $\mathfrak{su}(\frac{n}{2})$ ,  $\mathfrak{sp}(\frac{n}{4})$ ,  $G_2 \subset \mathfrak{so}(7)$  and  $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$  do not admit nonzero invariant symmetric 3-tensors. We claim that in each case the module  $\odot^2 \mathbb{R}^n$  does not contain any submodule isomorphic to  $\mathbb{R}^n$ . To check this it is enough to pass to the complexifications and use tables from [23], where one may find decompositions of the modules  $\odot^2 \mathbb{C}^n$  for all considered representations. This proves the claim for the Lie algebras  $\mathfrak{so}(n)$ ,  $G_2$  and  $\mathfrak{spin}(7)$ . For the proof of the claim for the Lie algebras  $\mathfrak{su}(\frac{n}{2})$  and  $\mathfrak{sp}(\frac{n}{4})$  also the decomposition (7.2) should be used (and this will imply the proof of the claim for the Lie algebras  $\mathfrak{u}(\frac{n}{2})$  and  $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$ ).  $\square$

Since  $T^{pkij} \neq 0$ , we conclude that  $\mathfrak{h} \subset \mathfrak{so}(n)$  is a symmetric Berger algebra.

From Lemma 5 it follows that the curvature tensor of  $(M, g)$  satisfies

$$(8.2) \quad R(e_k, q) \cdot R = \nabla_{e_k; q}^2 R - \nabla_{q; e_k}^2 R = 2T^{pkij}(p \wedge e_i) \odot (p \wedge e_j).$$

On the other hand, we write  $R = R_0 + P + T$  as in Section 4 (for  $\mathfrak{g}$  under consideration it holds  $L = v = 0$ ), and we get

$$R(e_k, q) \cdot R = P(e_k, q) \cdot (R_0 + P + T) + T(e_k, q) \cdot (R_0 + P + T).$$

It can be directly checked that

$$T(e_k, q) \cdot R_0 = 8T^{ik}\delta_{it}R_0^{jlrt}(e_j \wedge e_l) \odot (p \wedge e_r).$$

Since the action of  $P(e_k, q)$  preserves the grading of  $V$ , we get

$$P(e_k, q) \cdot P + T(e_k, q) \cdot R_0 = 0,$$

which can be rewritten in the form

$$(P(e_k, q) \cdot P)(Y, q) + 8R_0(T^{ik}e_i, Y), \quad Y \in E.$$

For each symmetric Berger algebra  $\mathfrak{h} \subset \mathfrak{so}(n)$ , the space  $\mathcal{R}(\mathfrak{h})$  is one-dimensional and is spanned by a tensor  $\mathcal{R}_0$ ; next,  $\mathcal{P}(\mathfrak{h}) \simeq \mathbb{R}^n$ , and each  $P \in \mathcal{P}(\mathfrak{h})$  is of the form  $P = R_0(\cdot, X)$  for some  $X \in \mathbb{R}^n$  [12].

Consequently, in our situation, if  $R_0$  is nonzero on an open set  $U$ , then there exists a section  $X$  of  $E$  over  $U$  such that  $P(Y, q) = R_0(Y, X)$  for all sections  $Y$  of  $E$  over  $U$ . In [10] it is shown that if we consider the new vector field

$$q' = -\frac{1}{2}g(X, X)p + X + q,$$

and the corresponding distribution  $E'$  with the sections

$$Y' = -g(Y, X)p + Y, \quad Y \in E,$$

then in new notations it holds  $P = 0$ . Next,  $R_0(T^{ik}e_i, \cdot) = 0$ . From the Bianchi identity it follows that  $R_0(Y, Z)T^{ik}e_i = 0$  for all  $Y, Z \in E$ . Since the values of  $R_0(Y, Z)$  at the point  $x$  generate  $\mathfrak{h}$ , and  $\mathfrak{h} \subset \mathfrak{so}(n)$  is irreducible, we get that  $T^{ik} = 0$ . This implies that  $R(e_k, q) \cdot R = 0$ , i.e.  $T^{pijk} = 0$ , and we get a contradiction.

Suppose that  $R_0 = 0$ . Then  $P(e_k, q) \cdot P = 0$ . This implies  $P = 0$  [12]. We get  $R = T$  is the curvature tensor of a pp-wave, hence  $(M, g)$  is a pp-wave and this is a contradiction.

**Case 2.**  $\mathfrak{h} \subset \mathfrak{so}(n)$  is not irreducible. Then we have the decomposition (2.2). From Lemma 1 it follows that each metric  $g_\alpha$ ,  $\alpha = 1, \dots, r$ , satisfies  $\nabla^3 R = 0$ . If the metric  $g_\alpha$  is indecomposable, then its holonomy algebra is  $\mathfrak{h}_\alpha + p \wedge E_\alpha$ . According to the previous case, this is impossible. Consequently the metric is decomposable, i.e. it is the sum of a Riemannian metric and of a Lorentzian metric. From Lemma 1 it follows that the holonomy algebra of the Riemannian part is isomorphic to  $\mathfrak{h}_\alpha$ . Hence, the Lorentzian part is of dimension 2 and its holonomy algebra is either trivial or it is isomorphic to  $\mathfrak{so}(1, 1)$ ; since there exists a null parallel vector field, the Lorentzian

part is flat. We conclude that there exists a null parallel vector field not proportional to  $p$ . Clearly, it should be of the form

$$q_\alpha = -\frac{1}{2}g(X_\alpha, X_\alpha)p + X_\alpha + q, \quad X_\alpha \in E_\alpha.$$

It holds  $R_\alpha(q_\alpha, \cdot) = 0$ . Hence,  $P_\alpha(Y, q) = -R_{\alpha 0}(Y, X_\alpha)$  for all  $Y \in E_\alpha$ . Considering as above the new vector field

$$q' = -\frac{1}{2}g(X, X)p + X + q, \quad X = -(X_1 + \dots + X_r),$$

we will get  $P = 0$ . The rest of the proof as in the previous case. The theorem is proved.  $\square$

### 9. Proof of Theorem 1

We know now that the holonomy algebra of the manifold  $(M, g)$  is  $p \wedge E$ , i.e. the manifold is a pp-wave and locally it is given by the metric

$$(9.1) \quad g = 2dudv + \sum_{i=1}^n (dx^i)^2 + Hdu^2,$$

where  $H$  is a function of  $x^i$  and  $u$ , see e.g. [9, Sect. 5.4]. We need only to decide for which functions  $H$  the metric is third-order symmetric. All tensors will be considered as contravariant. It can be shown that with respect to the frame  $p = \partial_v, e_i = \partial_{x^i}, q = \partial_u - \frac{1}{2}\partial_v$  it holds

$$\nabla p = 0, \quad \nabla e_i = \frac{1}{2}H_{,i}p \otimes p, \quad \nabla q = -\sum_i \frac{1}{2}H_{,i}p \otimes e_i,$$

where the comma denotes the partial derivative. This implies [2]

$$\begin{aligned} R &= \frac{1}{2} \sum_{i,j} H_{,ij} (p \wedge e_i) \odot (p \wedge e_j), \quad \nabla R = \frac{1}{2} \sum_{i,j} \left( \sum_k H_{,ijk} e_k \otimes + H_{,iju} p \right) \otimes (p \wedge e_i) \otimes (p \wedge e_j), \\ \nabla^2 R &= \sum_{i,j} \left( \left( \frac{1}{2} H_{,ijuu} - \frac{1}{4} \sum_k H_{,k} H_{,ijk} \right) p \otimes p \right. \\ &\quad \left. + \frac{1}{2} \sum_k H_{,ijk} (p \odot e_k) + \frac{1}{2} \sum_{k,l} H_{,ijkl} (e_k \otimes e_l) \right) \otimes (p \wedge e_i) \odot (p \wedge e_j). \end{aligned}$$

From Lemma 5 it follows that  $H_{,ijkl} = H_{,ijk}u = 0$ . Consequently,

$$H = H_{ijk}x^i x^j x^k + F_{ij}(u)x^i x^j + G_i(u)x^i + K(u), \quad H_{ijk} \in \mathbb{R}.$$

Next,

$$\nabla^2 R = \sum_{i,j} \left( \frac{1}{2} H_{,ijuu} - \frac{1}{4} \sum_k H_{,k} H_{,ijk} \right) p \otimes p \otimes (p \wedge e_i) \odot (p \wedge e_j),$$

and the equality  $\nabla^3 R = 0$  is equivalent to the equations

$$\partial_{x^i} (2H_{,ijuu} - \sum_k H_{,k} H_{,ijk}) = 0, \quad \partial_u (2H_{,ijuu} - \sum_k H_{,k} H_{,ijk}) = 0.$$

The first equation implies  $\sum_k H_{,kl} H_{,ijk} = 0$ . Applying  $\partial_{x^s}$ , we get  $\sum_k H_{,kls} H_{,ijk} = 0$ . For fixed  $l$  and  $i$  this equality means that the squire of the symmetric matrix  $(H_{,ijk})_{j,k=1}^n$  is zero, consequently,  $H_{,ijk} = 0$ . We are left with the equation  $H_{,ijuu} = 0$ . This implies that

$$F_{ij}(u) = H_{2ij}u^2 + H_{1ij}u + H_{0ij}.$$

The obtained metric has the same curvature tensor as the metric from the formulation of the theorem, this implies that the both metrics are isometric.  $\square$

**Conclusion.** We have extended and improved the methods from [2] in order to classify the third-order symmetric Lorentzian manifolds. It is natural to ask about the classification of the  $k$ -th order symmetric Lorentzian manifolds for arbitrary  $k \geq 4$  [4, 18, 19]. The description of the holonomy-invariant tensors in  $\otimes^{k-1}V \otimes \mathcal{R}(\mathfrak{g})$  becomes much more complicated. This means that

the algebraic approach developed by us must be intensively combined with geometric and analytic approaches from [22, 5]. It is not enough to use the fact that the tensor  $\nabla^{k-1}R$  is holonomy-invariant, but also the fact that this is the  $(k-1)$ -th covariant derivative of the curvature tensor must be applied. Formulas similar to (8.2) should be intensively used.

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